Rotating traversable wormholes

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Abstract

The general form of a stationary, axially symmetric traversable wormhole is discussed.
This provides an explicit class of rotating wormholes that generalize the static, spherically
symmetric ones first considered by Morris and Thorne. In agreement with general analyses,
it is verified that such a wormhole generically violates the null energy condition at the
throat. However, for suitable model wormholes, there can be classes of geodesics falling
through it which do not encounter any energy-condition-violating matter. The possible
presence of an ergoregion surrounding the throat is also noted.

1. Introduction

The concept of a *traversable wormhole* was first put forward by Morris and Thorne [1] in 1988. Unlike those previously considered, such as the Einstein–Rosen bridge [2] or the microscopic charge-carrying wormholes of Wheeler [3], traversable wormholes by definition permit the two-way travel of objects like human beings. Despite the dubious possibility of ever creating or finding such a wormhole, their study has opened up remarkably fruitful avenues of research. These include the fundamental properties of such wormholes [4–6], their use as time-machines [7,8] and the associated problems of causality violation [9–11], as well as the structure of quantum or Planck-scale wormholes [12,13].

Perhaps the key feature in the analysis of Morris and Thorne [1], is that they first list the conditions that a traversable wormhole must satisfy, and then use the Einstein equations to deduce the form of the matter required to maintain the wormhole. This is opposite to the usual procedure of postulating the matter content, and then solving the Einstein equations to obtain the space-time geometry (a step which is often very difficult, if not impossible). The paradigm shift enables a surprising amount of information to be deduced about such wormholes.

They considered a static, spherically symmetric and asymptotically flat space-time with the metric

$$\mathrm{d}s^2 = -\mathrm{e}^{\Phi(r)}\mathrm{d}t^2 + \left(1 - \frac{b(r)}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right),\tag{1}$$

where Φ and b are two arbitrary functions of r known as the redshift and shape functions respectively. The former determines the gravitational redshift of an infalling object, while the latter characterizes the shape of the wormhole as seen using an embedding diagram; hence their names. It was shown in Ref. [1] that for this metric to describe a wormhole, b must satisfy a certain flare-out condition, in which case (1) describes two identical asymptotic universes joined together at the 'throat' $r = b$. The condition that the wormhole be traversable, in particular, means that there are no event horizons or curvature singularities. This translates to the requirement that Φ be finite everywhere.

Morris and Thorne [1] then went on to prove that the metric (1), together with the wormhole-shaping and traversality conditions on b and Φ, imply that the corresponding stress-energy tensor necessarily violates the null (and therefore also the weak) energy condition [14]. They called this form of matter 'exotic', an acknowledgement of the fact that

there is an astronomical, perhaps impossible, price to be paid for interstellar travel using wormholes.

This has not prevented some authors from studying other classes of traversable wormholes, in the hope of minimizing the violation of the energy conditions. It was realized early on that by giving up spherical symmetry, it is possible to move the exotic matter around in space so that some observers falling through the wormhole would not encounter it [15]. This was demonstrated by cutting out holes in Minkowski space, and joining them up with a thin (delta-function) layer of matter.

Another natural generalization of the work of Morris and Thorne is to include time dependence. Perhaps the simplest way is to include a time-dependent conformal factor in the metric (1), while preserving spherical symmetry [16–18]:

$$\mathrm{d}s^2 = \Omega^2(t) \left[-\mathrm{e}^{\Phi(r)}\mathrm{d}t^2 + \left(1 - \frac{b(r)}{r} \right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2 \right) \right]. \qquad (2)$$

When $\Omega(t)$ is increasing, this metric represents a conformally expanding Morris–Thorne wormhole. It has been proposed that this might describe a wormhole being 'pulled out' of the space-time foam to a macroscopic size during the inflationary epoch [16]. If the wormhole were expanding fast enough, it appears possible to avoid any violation of the energy conditions. But as explained in Ref. [18], this is because any observer travelling through the wormhole would see its 'radius' increasing all the way, and thus it would not qualify as a wormhole in the usual sense.

In this paper, I shall construct the stationary and axially symmetric generalization of the Morris–Thorne wormhole (1). This would physically describe rotating wormholes. There are a few reasons why considering this case is important: it is perhaps the most general extension of the Morris–Thorne wormhole that one can fruitfully consider, short of a system with no space-time symmetries (which would be quite impractical to analyse with present techniques). This could then be used to derive or model explicit wormhole solutions, both classical [19] and semi-classical [13], that are of interest in various contexts. Of course, another reason is that if an arbitrarily advanced civilization [7] were to create a traversable wormhole, it would in all likelihood be aspherical or rotating. This would enable would-be interstellar travelers to avoid passing through and interacting with the exotic matter needed to maintain the wormhole.

The philosophy adopted in this paper will be the same as that of Ref. [1]. I shall begin by writing down the most general metric respecting the above-mentioned symmetries, and

examine the conditions under which it would describe a traversable wormhole. It is then explicitly shown that such a wormhole would, unfortunately but inevitably, violate the null energy condition for a class of null vectors at its throat. Finally, a concrete example of a rotating wormhole is used to demonstrate the avoidance of the exotic matter by certain observers, and the possible presence of an ergoregion surrounding the throat.

2. Canonical form of the metric

The space-times that we are interested in will be stationary and axially symmetric. The former means the space-time possesses a time-like Killing vector field $\xi^a \equiv (\partial/\partial t)^a$ generating invariant time translations, while the latter means it has a space-like Killing vector field $\psi^a \equiv (\partial/\partial\varphi)^a$ generating invariant rotations with respect to the angular coordinate φ. It is well-known, from the work of Papapetrou and Carter, that the most general stationary and axisymmetric metric can be written as [20–22]

$$\mathrm{d}s^2 = g_{tt}\mathrm{d}t^2 + 2g_{t\varphi}\mathrm{d}t\mathrm{d}\varphi + g_{\varphi\varphi}\mathrm{d}\varphi^2 + g_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{3}$$

where the indices $i, j = 1, 2$ run over the two remaining coordinates. This metric is uniquely determined up to coordinate transformations of (x^1, x^2). Such transformations can be used to adapt the metric for specific problems. For example, the choice $g_{11} = g_{22}$ and $g_{12} = 0$ turns it into a form which enables the Einstein equations to be simplified considerably (see, e.g., Ref. [23]).

We shall, however, use this freedom to cast the metric (3) into spherical polar coordinates by setting $g_{22} = g_{\varphi\varphi}/\sin^2 x^2$ [24]:

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \mathrm{e}^\mu\mathrm{d}r^2 + r^2K^2\left[\mathrm{d}\theta^2 + \sin^2\theta(\mathrm{d}\varphi - \omega\mathrm{d}t)^2\right], \tag{4}$$

where the four gravitational potentials N, μ, K and ω depend on $(x^1, x^2) \equiv (r, \theta)$ only. This form of the metric has the distinct advantage of making the physics transparent. (It was first used by Hartle [25,26] in the study of relativistic rotating stars.) The quantity $\omega(r, \theta)$ is the angular velocity $\mathrm{d}\varphi/\mathrm{d}t$ acquired by a particle that falls freely from infinity to the point (r, θ), and which gives rise to the well-known dragging of inertial frames or Lense–Thirring effect in general relativity. $K(r, \theta)$ is a positive, nondecreasing function of r that determines the 'proper radial distance' R measured at (r, θ) from the origin:

$$R \equiv rK, \qquad R_r > 0. \tag{5}$$

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($R_r \equiv \partial R / \partial r$, etc.) Notice that $2\pi R \sin\theta$ can be geometrically interpreted as the proper circumference of the circle located at coordinate values (r, θ), with φ ranging from 0 to 2π.

The discriminant [22] of the metric (4) is

$$D^2 \equiv -g_{tt}g_{\varphi\varphi} + g_{t\varphi}{}^2 = (NKr\sin\theta)^2, \tag{6}$$

which implies that an event horizon appears whenever $N = 0$ (see, e.g., Sec. 7.10 of Ref. [27]). To ensure that the metric is nonsingular on the rotation axis $\theta = 0, \pi$, the usual regularity conditions on N, μ and K have to be imposed. Essentially, this means their θ derivatives have to vanish on the rotation axis.

To turn the stationary, axisymmetric metric (4) into a form suitable for describing a traversable wormhole, we write

$$\mu(r,\theta) = -\ln\left(1 - \frac{b(r,\theta)}{r}\right), \tag{7}$$

in terms of the new function $b(r,\theta)$. It clearly reduces to the Morris–Thorne canonical metric (1) in the limit of zero rotation and spherical symmetry:

$$N(r,\theta) \to e^{\Phi(r)}, \qquad b(r,\theta) \to b(r), \qquad K(r,\theta) \to 1, \qquad \omega(r,\theta) \to 0. \tag{8}$$

In analogy with that case, we shall only consider the coordinate range $r \geq b$ and identify the apparent singularity at $r = b \geq 0$ with the throat of the wormhole. N, K and ω are assumed to be otherwise well-behaved at the throat. Now, it is readily checked that the scalar curvature of the space-time (4) has the following terms of order $(r - b)^{-2}$:

$$-\frac{1}{(rK)^2}\left(\mu_{\theta\theta} + \frac{1}{2}\mu_\theta{}^2\right) = -\frac{3}{2}\frac{1}{(rK)^2}\frac{b_\theta{}^2}{(r-b)^2}. \tag{9}$$

If the throat is to be free of curvature singularities, b_θ has to vanish there. Hence, the throat is located at some constant value of r. (Note that this does not mean the throat is spherically symmetric, since the proper radial distance R in general still has a θ dependence coming from K.)

Now, in order for the geometry (4) to have the shape of a wormhole, b must satisfy a so-called flare-out condition when $r = b$. This can be seen by embedding it in a higher-dimensional space, following Ref. [1]. For constant t and θ, (4) becomes

$$ds^2 = \left(1 - \frac{b(r)}{r}\right)^{-1}dr^2 + r^2K^2\sin^2\theta d\varphi^2$$
$$= \left(1 - \frac{\beta(\rho)}{\rho}\right)^{-1}d\rho^2 + \rho^2 d\varphi^2, \tag{10}$$

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where $\rho \equiv R \sin\theta$, and β is defined correspondingly in terms of b. The throat is at $\rho = \beta$. We shall embed this two-surface in an (unphysical) three-dimensional Euclidean space, which has the metric

$$\mathrm{d}\tilde{s}^2 = \mathrm{d}z^2 + \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\varphi^2, \tag{11}$$

in cylindrical coordinates. This surface is then described by the function $z = z(\rho)$, which satisfies

$$\frac{\mathrm{d}z}{\mathrm{d}\rho} = \pm \left(\frac{\rho}{\beta(\rho)} - 1 \right)^{-1/2}. \tag{12}$$

That it has the characteristic shape of a wormhole, as illustrated in Figs. 1 and 2 of Ref. [1], means the flare-out condition $\mathrm{d}^2\rho/\mathrm{d}z^2 > 0$ must be satisfied at the throat. But, we have

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}z^2} = \frac{\mathrm{d}\rho}{\mathrm{d}r} \frac{\mathrm{d}^2 r}{\mathrm{d}z^2}, \tag{13}$$

when $r = b$. Since $\mathrm{d}\rho/\mathrm{d}r$ is positive by (5), the flare-out condition is equivalent to

$$\frac{\mathrm{d}^2 r}{\mathrm{d}z^2} = \frac{b - b_r r}{2b^2} > 0, \tag{14}$$

at the throat. This is precisely the same condition as in the Morris–Thorne wormhole [1].

Hence, the form of $b(r, \theta)$ in g_{11} is very similar to its counterpart in the Morris–Thorne wormhole (1). Recall that it is possible to define a new radial coordinate l, in that case, by

$$\frac{\mathrm{d}l}{\mathrm{d}r} \equiv \pm \left(1 - \frac{b}{r} \right)^{-1/2}, \tag{15}$$

which is well-behaved across the wormhole throat. In the present case, it is also possible to make this change of coordinate in the immediate vicinity of the throat, where $b_\theta = 0$ as we observed above. The metric (4) then becomes

$$\mathrm{d}s^2 = -N^2(l, \theta)\mathrm{d}t^2 + \mathrm{d}l^2 + r^2(l)K^2(l, \theta)\left[\mathrm{d}\theta^2 + \sin^2\theta(\mathrm{d}\varphi - \omega(l, \theta)\mathrm{d}t)^2\right], \tag{16}$$

to first order in $r - r_0$, where r_0 is the location of the throat. This metric smoothly connects the two asymptotic regions of the space-time across the throat, unlike (4) which is singular there. Of course, if $b = b(r)$ is independent of θ, then l defined by (15) is valid everywhere and takes the range $(-\infty, \infty)$. The metric (16) then covers the entire space-time. Without loss of generality, we may assume the wormhole throat is at $l = 0$, so that l is positive on one side of the throat and negative on the other. The asymptotic regions are at $l = \pm\infty$.

Although it is implicit in the form of the above metric that the two regions connected by the wormhole are isometric, it is possible to generalize it to the case when they are not. The static, spherically symmetric case was discussed in Sec. 11.2 of Ref. [14]; and the same methods apply here. One simply introduces different gravitational potentials, $N_\pm$, $b_\pm$, $K_\pm$, $\omega_\pm$, for each of the regions labeled by $\pm$. However, they have to match up appropriately at the throat, to ensure that the metric components (in terms of l) are continuous and differentiable there.

We also require that the metric (4) be asymptotically flat, in which case

$$N \to 1, \qquad \frac{b}{r} \to 0, \qquad K \to 1, \qquad \omega \to 0, \tag{17}$$

as $r \to \infty$. Thus, r is asymptotically the proper radial distance. In particular, if

$$\omega = \frac{2a}{r^3} + \mathrm{O}\left(\frac{1}{r^4}\right), \tag{18}$$

then by changing to Cartesian coordinates, it can be checked that a is the total angular momentum of the wormhole. Its mass and charge, if any, can also be deduced in the usual manner [14].

To summarize, the canonical metric for a stationary, axisymmetric traversable wormhole can be written as

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \left(1 - \frac{b}{r}\right)^{-1}\mathrm{d}r^2 + r^2 K^2 \left[\mathrm{d}\theta^2 + \sin^2\theta(\mathrm{d}\varphi - \omega\mathrm{d}t)^2\right], \tag{19}$$

where N, b, K and ω are functions of r, and of θ such that it is regular on the symmetry axis $\theta = 0, \pi$. It describes two identical, asymptotically flat regions joined together at the throat $r = b > 0$. N is the analog of the redshift function in (1), and it has to be finite and nonzero to ensure that there are no event horizons or curvature singularities. b is the shape function which satisfies $b \leq r$. At the throat itself, it has to be independent of θ, i.e., $b_\theta = 0$, and obey the flare-out condition $b_r < 1$. K determines the proper radial distance as in (5), while ω governs the angular velocity of the wormhole.

The space-time (19) will, in general, have nonvanishing stress-energy tensor components T_{tt}, $T_{t\varphi}$ and $T_{\varphi\varphi}$, as well as T_{ij}. They have the usual physical interpretations; in particular, $T_{t\varphi}$ characterizes the rotation of the matter distribution. For a static, spherically symmetric wormhole, it turns out that the matter required to support it must have a radial tension at the throat exceeding its mass-energy density [1]. This signifies a breakdown of the celebrated energy conditions of general relativity [28]. As we shall now see, the latter is also unavoidable for the class of wormholes described by (19).

3. Violation of the null energy condition

It can be shown, under very general conditions, that a traversable wormhole violates the averaged null energy condition in the region of the throat [5,6], by using the Raychaudhuri equation [28] together with the fact that a wormhole throat by definition defocuses light rays. However, it is often useful to carry out a specific analysis, as in the present case. In this section, I shall explicitly show that the null energy condition [28,14]

$$R_{ab}k^a k^b \geq 0 \,, \tag{20}$$

where R_{ab} is the Ricci tensor of the space-time (19), is violated by a class of null vectors k^a at the throat. This would allow us to determine the precise location of the violation, and identify the gravitational potentials responsible for it.

Recall that for the Morris–Thorne wormhole (1), the null vectors in question are radial ones, which can be taken to be $k^a = (\sqrt{-g^{tt}}, \pm\sqrt{g^{11}}, 0, 0)$ [1,14]. This choice is also possible for the stationary, axisymmetric case (19), provided $g_{tt} < 0$ everywhere. But the latter is not always true in a rotating system, as we shall see in the following section. A more natural choice turns out to be the following null vector:

$$k^a = \left(\frac{1}{N}, -\mathrm{e}^{-\mu/2}, 0, \frac{\omega}{N}\right), \tag{21}$$

where we may take N to be positive, without loss of generality. In the asymptotic region, $k^a = (1, -1, 0, 0)$, so it represents the 4-velocity of a (null) particle that is directed radially inwards. But it acquires a nonzero angular velocity ω as the throat is approached, due to the dragging of inertial frames [24] by the wormhole. We have, at the throat itself,

$$R_{ab}k^a k^b = \mathrm{e}^{-\mu}\mu_r \frac{(rK)_r}{rK} - \frac{\omega_\theta^2 \sin^2\theta}{2N^2} - \frac{1}{4}\frac{\mu_\theta^2}{(rK)^2} - \frac{1}{2}\frac{(\mu_\theta \sin\theta)_\theta}{(rK)^2 \sin\theta} + \frac{(N_\theta \sin\theta)_\theta}{(rK)^2 N \sin\theta} \,. \tag{22}$$

But by the flare-out condition (14),

$$\mathrm{e}^{-\mu}\mu_r = \frac{1}{r^2}(b_r r - b) < 0 \,. \tag{23}$$

Furthermore, $R = rK$ is a monotonically increasing function of r. Thus, the first term on the right-hand side of (22) is negative-definite. The second term is also manifestly nonpositive.

It remains to show that the sum of the remaining three terms is negative at some point $\theta \in [0, \pi]$ on the throat (excluding the trivial case when they are identically zero). We first rewrite them as

$$\left(f_1{}^2 - f_2{}^2\right) + \frac{(f \sin \theta)_\theta}{\sin \theta} , \tag{24}$$

multiplied by an overall positive factor of $(rK)^{-2}$, where

$$f_1 \equiv (\ln N)_\theta , \qquad f_2 \equiv \tfrac{1}{2}\mu_\theta , \qquad f \equiv f_1 - f_2 . \tag{25}$$

Note that the regularity conditions on N and μ imply that f vanishes at $\theta = 0$ and π. Now, suppose $f < 0$ at some point $\theta \in (0, \pi)$. By continuity, there is an interval in $(0, \pi)$ such that $f \sin \theta < 0$ and $(f \sin \theta)_\theta < 0$. The former means that $f_1{}^2 < f_2{}^2$, and hence (24) is negative in this interval. On the other hand, suppose $f \geq 0$ everywhere in $[0, \pi]$. Then, as $\theta \to \pi$, we have $f \to 0$ and $(f \sin \theta)_\theta \to 0^-$. The former means that the first term of (24) vanishes in this limit, while the latter implies that the second term of (24) is negative in this limit. Hence, we have proved that the right-hand side of (22) is negative at some point on the throat, and there is consequently a violation of the null energy condition there.

A similar argument shows that the sum of the last three terms on the right-hand side of (22) is positive at some point in the interval $(0, \pi)$. By choosing N and μ appropriately, it is possible to render $R_{ab}k^a k^b$ positive at this point. Thus, the exotic matter supporting the wormhole can be moved around the throat, so that some class of infalling observers would not encounter it; an example is discussed in the following section. This is to be contrasted with the static, spherically symmetric case, where all such observers will experience a violation of the energy condition. But the key point in the above result is that one can never avoid the use of exotic matter altogether.

4. Example of a rotating wormhole

In the absence of any further physical input, it is possible to construct an infinity of wormholes of the form (19), limited only by one's imagination. Morris and Thorne [1] have justifiably argued that the use of exotic matter should be minimized. This is readily achieved by confining the exotic matter to a small region around the throat, and surrounding it with ordinary matter. They have also put constraints on the wormholes to make them suitable for human travel — the so-called engineering considerations. However, we shall not be overly concerned with either of these issues. The purpose here is to

illustrate, by means of an explicit example, some general features of a rotating wormhole, rather than to construct a semi-realistic wormhole.

We shall take the gravitational potentials of (19) to be

$$N = K = 1 + \frac{(4a\cos\theta)^2}{r}, \qquad b = 1, \qquad \omega = \frac{2a}{r^3}. \tag{26}$$

Perhaps the first point to note is that a is the angular momentum of the resulting wormhole. When it vanishes, the metric reduces to the spherically symmetric, zero-tidal-force Schwarzschild wormhole of Ref. [1]. It is so-called because the embedded surface for the equatorial plane of the wormhole is given by (valid even for nonzero a, as can be checked using (12))

$$z(r) = \pm 2\sqrt{r - 1}, \tag{27}$$

which is identical to that obtained by embedding the Schwarzschild black hole space-time appropriately. The throat at $r = 1$ has proper radius $R = 1 + (4a\cos\theta)^2$. Thus, it has a dumbbell-like shape as in Fig. 1, with minimum radius at the equator. Also note that since b is independent of θ, we may introduce the new radial coordinate

$$l = \pm \left[\sqrt{r(r-1)} + \ln\left(\sqrt{r} + \sqrt{r-1}\right) \right], \tag{28}$$

satisfying (15), and the resulting metric given by (16) is well-behaved across the throat.

Let us consider geodesic motion in this space-time, which for simplicity, will be restricted to the equatorial plane $\theta = \pi/2$. The 4-velocity vector of the geodesic is then

$$k^a \equiv \dot{x}^a = (\dot{t}, \dot{r}, 0, \dot{\varphi}), \tag{29}$$

where an overdot denotes derivative with respect to the affine parameter along the geodesic. The time-like Killing vector field ξ^a and axial Killing vector field ψ^a respectively yield a conserved energy E, and angular momentum L, per unit rest mass for the geodesic (see, e.g., Ref. [29]):

$$E = -g_{ab}\xi^a k^b = -g_{tt}\dot{t} - g_{t\varphi}\dot{\varphi},$$
$$L = g_{ab}\psi^a k^b = g_{t\varphi}\dot{t} + g_{\varphi\varphi}\dot{\varphi}. \tag{30}$$

Furthermore, we have

$$g_{ab}k^a k^b = -\kappa, \tag{31}$$

where $\kappa = 0$ for null geodesics and $\kappa = 1$ for time-like ones.

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The pair of equations (30) can be solved to obtain

$$\dot{t} = \frac{1}{D^2}(g_{\varphi\varphi}E + g_{t\varphi}L), \qquad \dot{\varphi} = -\frac{1}{D^2}(g_{t\varphi}E + g_{tt}L), \tag{32}$$

where D^2 is given in (6). Substituting this result into (31) yields an expression for $\dot{r}$:

$$\dot{r}^2 = g^{11}\left[\frac{1}{D^2}(g_{\varphi\varphi}E^2 + 2g_{t\varphi}EL + g_{tt}L^2) - \kappa\right]. \tag{33}$$

Thus, a geodesic freely falling towards the wormhole without any angular momentum would have 4-velocity

$$k^a = \left(\frac{E}{N^2}, -\sqrt{e^{-\mu}\left(\frac{E^2}{N^2} - \kappa\right)}, 0, \frac{\omega E}{N^2}\right). \tag{34}$$

When $a = 1/4$ say, it can be verified that $R_{ab}k^a k^b = E^2/r^3$ for such geodesics which are null. This quantity is clearly positive. Furthermore, for time-like geodesics, we have

$$G_{ab}k^a k^b = \frac{1}{r^3}(E^2 - 3) + \frac{9}{16}\left(\frac{1}{r^7} - \frac{1}{r^6}\right), \tag{35}$$

where G_{ab} is the Einstein tensor of the space-time. When these geodesics have sufficiently high energy $E \gtrsim 1.75$, (35) is positive everywhere along the path. Hence, these time-like and null geodesics are able to traverse the wormhole without encountering any exotic matter having negative energy density $T_{ab}k^a k^b$. (After the throat is crossed, the minus sign of the k^1 component of (34) becomes a plus sign.)

If the rotation of the wormhole is sufficiently fast, g_{tt} becomes positive in some region outside the throat, indicating the presence of an ergoregion where particles can no longer remain stationary with respect to infinity. For the above example, this occurs when $r^2 = |2a\sin\theta| > 1$, i.e., when $|a| > 1/2$. Notice that the ergoregion does not completely surround the throat, but forms a 'tube' around the equatorial region as illustrated in Fig. 1. This is characteristic of traversable wormholes: the ergoregion would necessarily intersect an event horizon at the poles, but since the latter is ruled out by definition, the ergoregion cannot extend to the poles. (It is also for this reason that the more traditional name of 'ergosphere' is hardly appropriate here.)

When an ergoregion is present, it is possible to extract (rotational) energy from the wormhole by the Penrose process [30,31], which was originally proposed for the Kerr black hole. This process relies on the fact that time-like particles need not have positive energy
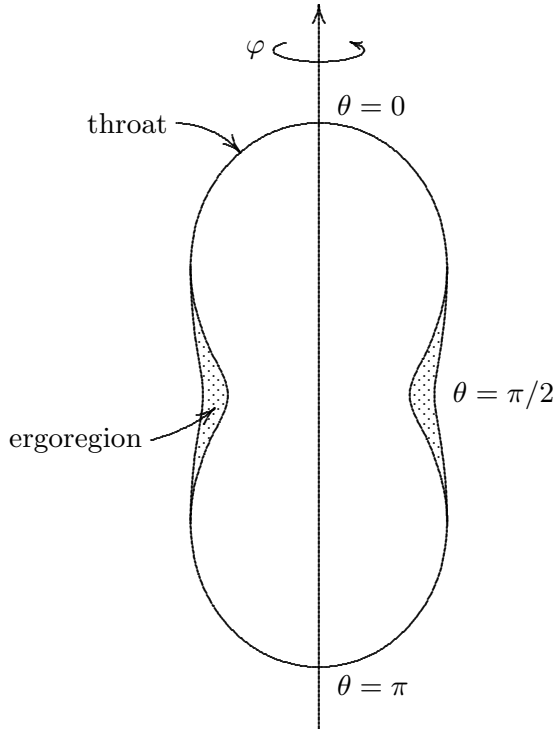
Fig. 1. Cross-sectional schematic of the wormhole throat. The shaded region indicates the ergoregion, if it is present, surrounding the throat at the equator.

in the ergoregion. Imagine an infalling particle breaking up into two inside the ergoregion. It is possible to arrange this breakup so that one of the resulting particles has negative total energy. The other particle can then travel out of the ergoregion along a geodesic, and would have more energy than what originally went in. However, the particle with negative energy would have an orbit confined entirely within the ergoregion. It is not possible for it to escape without gaining additional energy from some other source.

5. Concluding remarks

In this paper, I have presented the stationary, axisymmetric generalization of the Morris–Thorne wormhole. In particular, I have written down the canonical form of the metric for such a wormhole. This would allow one to describe rotating wormholes within a general framework. Although the null energy condition is generically violated at the throat, it is possible for geodesics falling through the wormhole to avoid this energy-condition-violating matter. Like rotating black holes, such wormholes can have ergoregions, from which energy can be extracted by the Penrose process.

Perhaps the most promising application of the results of this paper lies in the semi-classical or quantum regime. Unlike classical matter fields, it is well-known that quantum fields *do* violate the energy conditions [32]. Indeed, self-consistent wormhole solutions have been found in Ref. [13], in which the stress-energy tensor is that of a quantized scalar field [33]. These wormholes have throats with radii of order of the Planck length, and could serve as a model for space-time foam. The possibility of generalizing this to wormholes with (slow) rotation has recently been discussed in Ref. [34].

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