# Max-Planck-Institut
## für Mathematik
## in den Naturwissenschaften
## Leipzig


**The current status of quantum fields in curved spacetime (Transparencies of a talk given at DPG meeting, Ulm, 17 March 2004)**

by


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# The Current Status of Quantum Fields in Curved Spacetime

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These are the transparencies of an invited plenary lecture at the annual meeting of the Deutsche Physikalische Gesellschaft in Ulm, Germany, 14-18 March, 2004. The lecture was delivered by Rainer Verch on Wednesday, 18 March 2004. The lecture was announced as follows in the book of abstracts (Verhandlungen) of the meeting:

Title:

**The current status of quantum field theory in curved spacetime**

Presented by:

**Rainer Verch, MPI for Mathematics in the Sciences, Leipzig**

Abstract:

In this talk, I will report on the developments in quantum field theory in curved spacetime which, during the past 10 years, have led to impressive progress. These developments are centered around concepts like the microlocal spectrum condition, quantum energy inequalities and local general covariance. With the help of these concepts, it has been possible to formulate and complete the renormalization program of perturbative quantum field theory on generic spacetime backgrounds, and to arrive at strong structural theorems like PCT and the connection between spin and statistics. Furthermore, some insight into the qualitative behavior of semiclassical gravity has been gained, showing that the occurrence of exotic spacetime scenarios is suppressed by the dynamical stability of quantum fields.

## Quantum Field Theory and Gravitation

**Quantum Field Theory:** Describes structure of elementary particles at small scales, interaction processes with very high energy transfer, localized in space and time

relevant in the sub-microscopic domain
$\sim 10^{-19} m$

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**Gravitation:** Interaction with infinite range, effective for large aggregates of matter

relevant (dominating) in the macroscopic and cosmic domain
$\sim 2 \cdot 10^{26} m$

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**Combination:** Processes and states of matter in extreme situations, i.e.:
extreme amounts of energy or matter at very small length scales

e.g.,

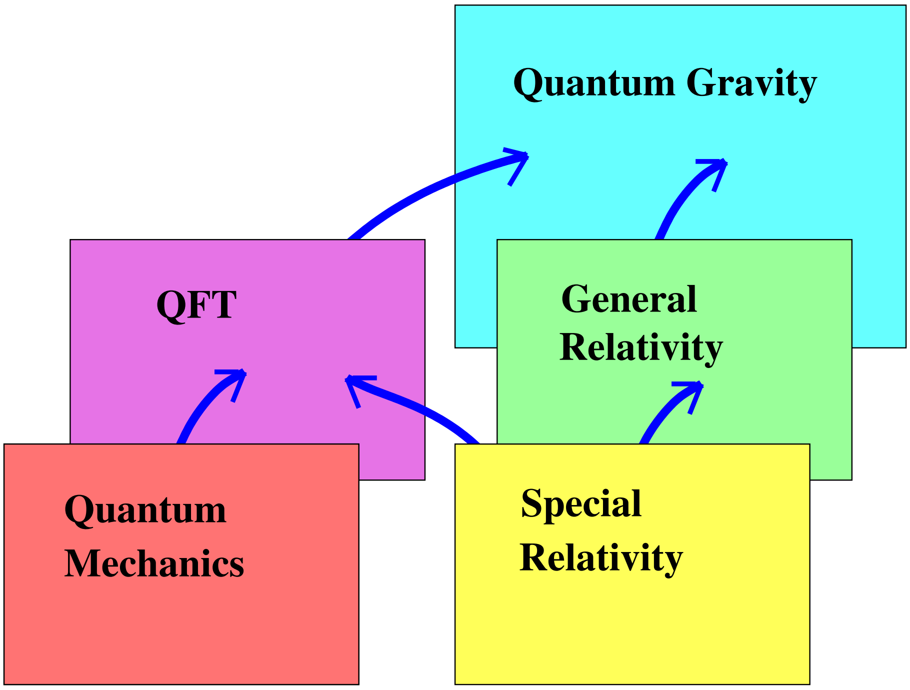
- collapse of a star to a black hole, space-time singularity in its interior
- Cosmology: Initial singularity of the Universe Planck scale, $\ell_P = 10^{-35} m$

**Generally expected**:

The current concepts of gravitation/general relativity and quantum physics are insufficient for an understanding of such phenomena, in particular:

Physical meaning of the singularity theorems needs to be clarified.

A theory of "quantum gravity" is needed which provides an extension (and unification) of general relativity, gravitation and quantum physics.




There are approaches to quantum gravity so far, most prominently:

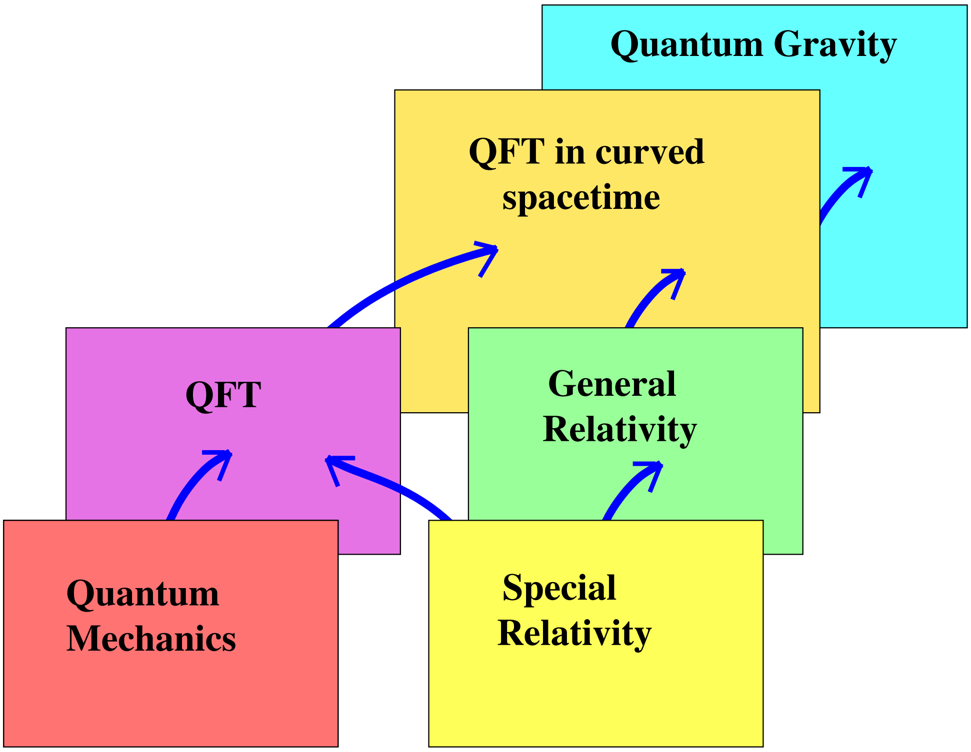
- Loop Quantum Gravity
- Noncommutative Geometry
- String/M Theory

...but still somewhat speculative.

An intermediate step on the route to quantum gravity:

**Quantum Field Theory in Curved Spacetimes**

and Semiclassical Gravity.




Quantum Field Theory in Curved Spacetimes means:

- microscopic (qft-) description of matter in outer gravitational fields,
- the structure of space and time remains classical (non-quantized), described in the sense of general relativity

**This approach leads to the following situation:**

♯ Spacetime is "curved", curvature induced by "outer" mass distribution $\leftrightarrow$ gravitational field

♯ Matter described by quantum fields propagating in the classical, curved "background" spacetime in the sense of "test fields", i.e.:

- *to first approximation*: Matter fields carry low mass-energy density, negligible as gravitational source

- *further step*: Consider mass-energy distribution of quantum matter fields as additional gravitational source; correction to outer gravitational fields (semiclassical gravity)

**Interesting effects**:

⋆ Hawking effect (thermal radiation by black holes)
⋆ Unruh effect
⋆ Particle creation in the early universe
⋆ Casimir effect (boundary effect)

# Basics of the Theory

## General Relativity:

Spacetime structure described by:

$$M \quad , \quad g_{\mu\nu}$$

4-dimensional manifold
"catalogue of events"

spacetime metric governs
propagation of light and
material particles

Einstein's field equations of gravity:

$$R_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) R(x) = -\frac{8\pi \mathrm{G}}{c^2} T_{\mu\nu}(x)$$

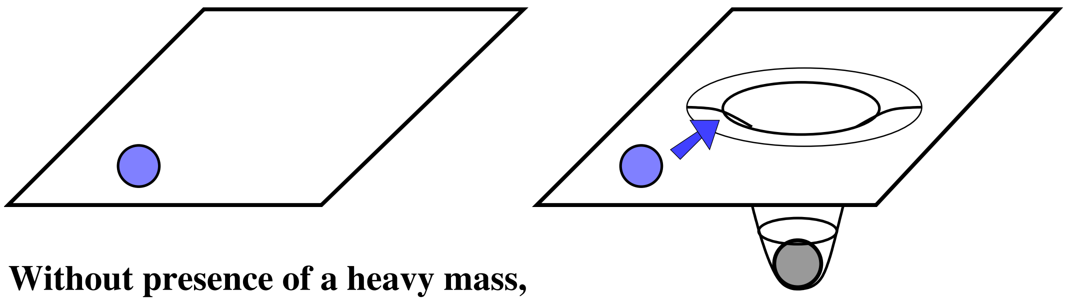
Curvature quantities of spacetime
metric $g_{\mu\nu}$, describe gravity

energy-momentum tensor of
matter distributed in spacetime

Fundamental principle of general relativity:

*"presence of energy/matter enforces spacetime curvature,
spacetime curvature governs the motion of matter"*

**Analogue: Iron ball on sheet of elastic material**



**Without presence of a heavy mass,
sheet = spacetime flat,
no curvature for blue ball
to follow**

**Presence of a "source mass"
inducing curvature,
blue ball follows curvature**

**Quantum Field Theory:**

A simple model:

Linear, scalar Klein-Gordon field
on a spacetime $M$ , $g_{\mu\nu}$

Field equation:

$$(\nabla^\mu \nabla_\mu + m^2)\varphi(x) = 0$$

**Advanced and retarded fundamental solutions**
(Green's functions)

$$G_+ \quad \text{and} \quad G_-$$

are uniquely determined.

**Quantization** through replacement

$$\underset{\text{number}}{\varphi(x)} \quad \longrightarrow \quad \underset{\text{operator-valued object}}{\Phi(x)}$$

with the properties

$$\begin{aligned} \Phi(x) &= \Phi(x)^* \quad \text{(hermiticity)} \\ (\nabla^\mu \nabla_\mu + m^2)\Phi(x) &= 0 \quad \text{(field equation)} \\ [\Phi(x), \Phi(y)] &= i\hbar(G_+(x,y) - G_-(x,y))\mathbf{1} \\ & \qquad \text{(commutation relations)} \end{aligned}$$

**Wanted:** Hilbert space representations of these algebraic relations, d.h.

$\Phi(x) \quad \longrightarrow$ operator (unbounded) in a Hilbert space

**Problem:**

There are many *different* such representations, each describing *different* physics!

Which is the correct one?

This is of particular relevance for the

**semiclassical Einstein equations**

$$R_{\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}(x)R(x) = -\frac{8\pi\mathrm{G}}{c^2}\left(T_{\mu\nu}(x) + \langle \mathbf{T}_{\mu\nu}(x)\rangle\right)$$

- $T_{\mu\nu}(x) =$ energy-momentum tensor of outer mass distribution

- $\langle \mathbf{T}_{\mu\nu}(x)\rangle =$ expectation value of energy-momentum tensor of $\Phi(x)$ in Hilbert space representation:

$$\langle \mathbf{T}_{\mu\nu}(x)\rangle \ = \langle\psi|\mathbf{T}_{\mu\nu}(x)|\psi\rangle$$

The global qualitative behaviour of solutions to the semiclassical Einstein equations depends crucially on the dynamical stability properties of the right hand side;

dynamical stability properties of $\langle \mathbf{T}_{\mu\nu}\rangle$ depend crucially on the Hilbert space representation of the quantum field operators.

In **special relativistic** quantum field theory:
Selection of Hilbert space representations through their dynamical behaviour with respect to spacetime symmetries.

Typical requirement:

For each "direction of time" $\vec{e}$ there should be a Hamilton operator $H(\vec{e})$ such that (in the Heisenberg picture)

$$\mathrm{e}^{iH(\vec{e})t}\Phi(x)\mathrm{e}^{-iH(\vec{e})t} = \Phi(x + t\vec{e})$$

with $\qquad H(\vec{e}) \geq 0 \qquad$ (**spectrum condition**),
and existence of a **vacuum vector** $\psi_0$ with

$$H(\vec{e})\psi_0 = 0$$

These conditions

- characterize "dynamical stability" of a quantum field theory in each Lorentz frame,
- are important for a clear-cut concept of particles,
- are independent of the particular quantum field model, i.e. independent of particle type or field equation,
- together with Poincaré covariance they allow important, model-independent conclusions about the general structure of quantum field theories, e.g.: PCT, spin and statistics and all that.

**To be desired:** For quantum fields on curved spacetimes, there should be a similar, general characterization of "dynamically stable" Hilbert space representations.

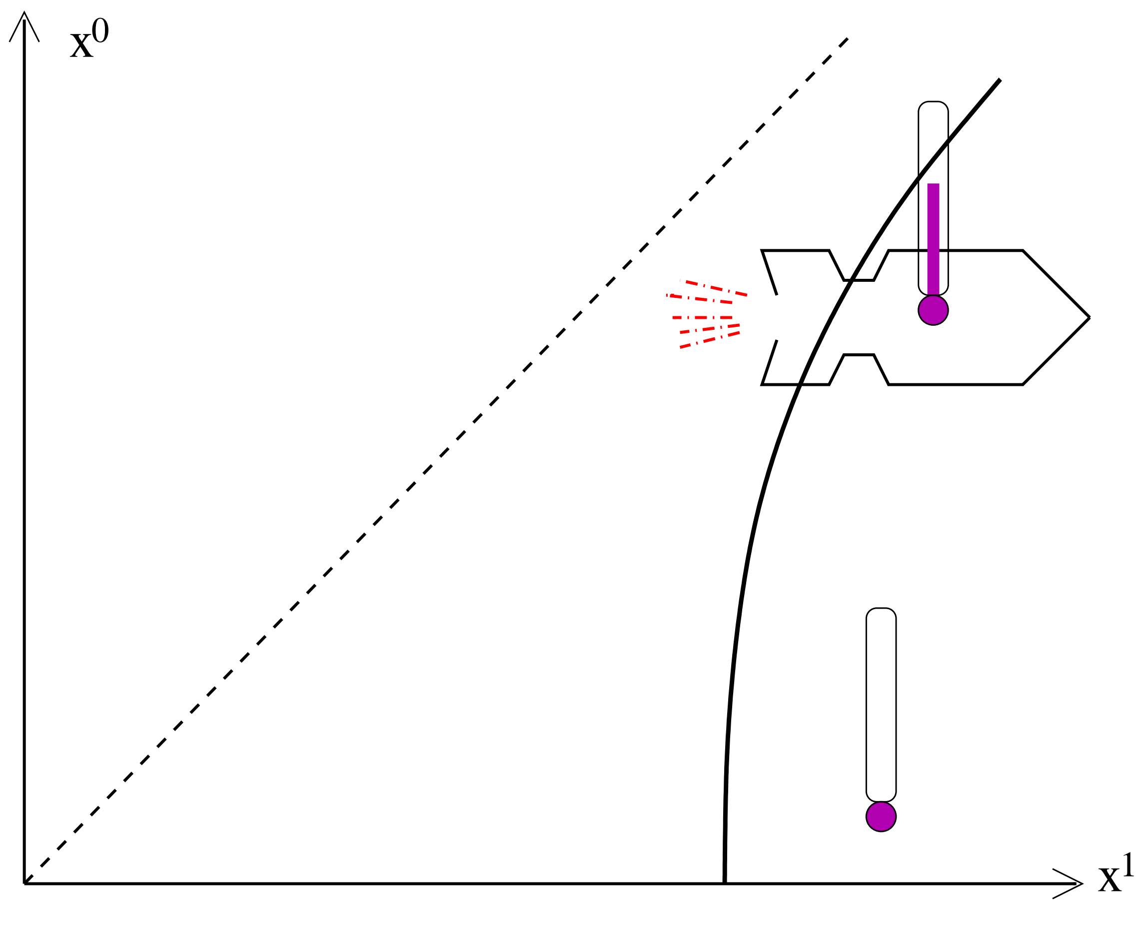
**But:** In the presence of spacetime curvature, there are no global inertial frames!

The concept of "particle" or "vacuum" becomes observer-dependent.

**Illustration:**

**The Fulling-Unruh Effect:** An observer moving with constant acceleration registers the vacuum state – defined with respect to a Lorentz frame – as a thermal equilibrium state having the temperature

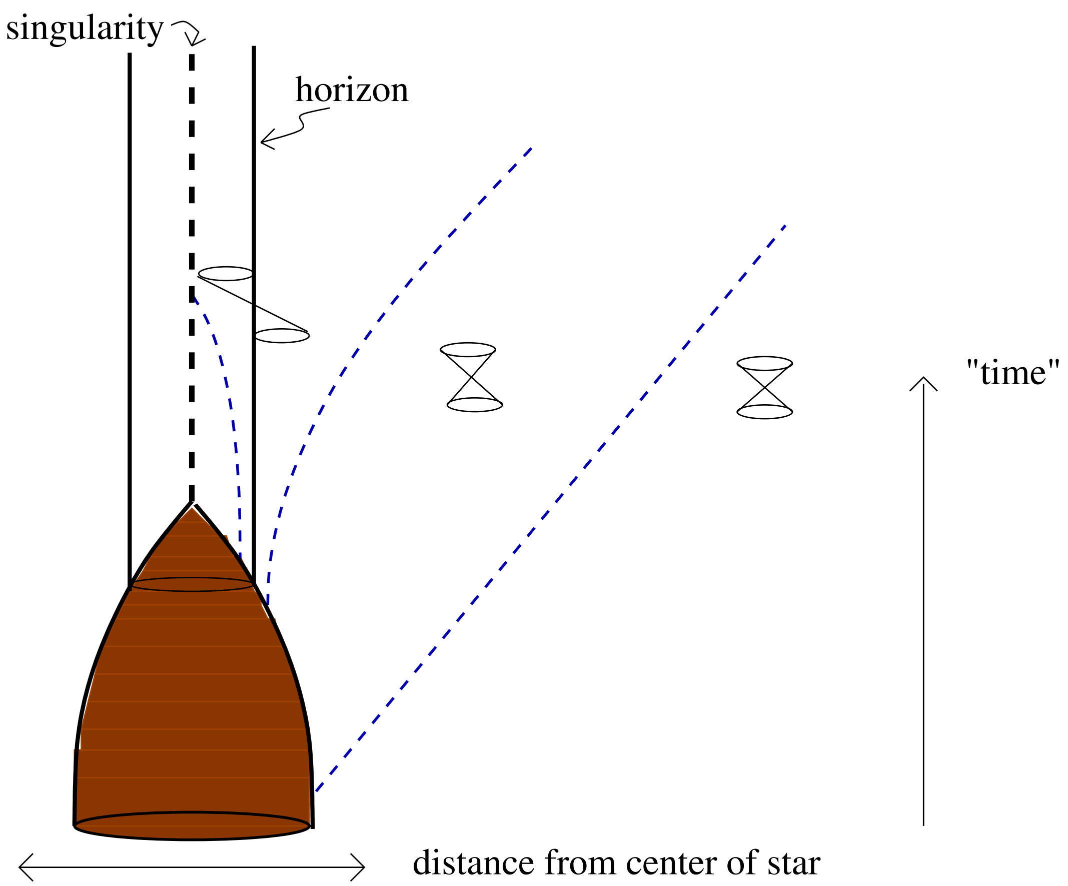
$$T_a = \frac{\hbar a}{2\pi k_B c}\,, \qquad a = \text{proper acceleration}$$



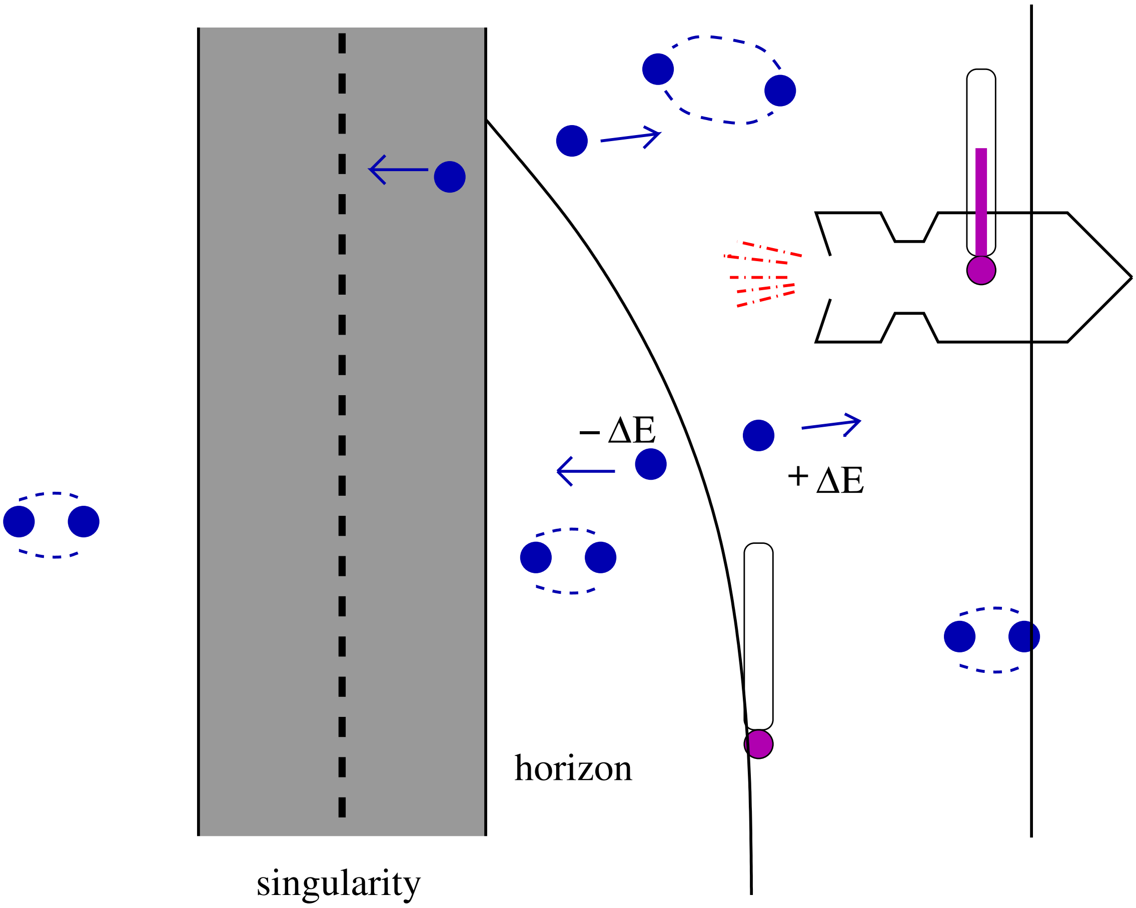
**Hawking Effect:** An observer kept (by acceleration) at constant distance to the black hole will register a thermal equilibrium state (at large times) having the temperature

$$T_{\text{Hawking}} = \frac{\hbar c^3}{8\pi G M} \, , \qquad M = \text{mass of black hole,}$$

if the quantum field state at early times (before the stellar collapse to a black hole) was a vacuum state.



**Hawking Effect** and analogy to the Fulling-Unruh Effect



## A Brief History of QFT in CST

**Pre–QFT period:**
Schrödinger 1932,
Dirac 1935,
Lichnerowicz 1960, DeWitt–Brehme 1961

**Early period:**
Takahashi–Umezawa 1957, Parker 1969
particle creation in an expanding universe

**Golden era (~1973–1978):**
Zeldovich
Hawking
Unruh
Fulling

starting with the advent of the Hawking effect, QFT in CST becomes a subject in its own right

**Phase of consolidation (~1978–1993):**
✗ Hadamard states
✗ Quantum energy inequalities
Scattering theory on black hole spacetimes, better understanding of Hawking effect

Dimock, Ford, Fredenhagen, Fulling, Haag, Kay, Sewell, Wald

**The past 10 years:**
✗ microlocal spectrum condition
<–> Hadamard condition
<–> Quantum energy inequalities
✗ Local renormalization programme
✗ Local general covariance
✗ Spin and statistics, PCT
✗ Causal regularity of solutions to semiclassical gravity

Brunetti, Fewster, Ford, Fredenhagen, Hollands, Kay, Junker, Moretti, Pfenning, Radzikowski, Roman, Sahlmann, Verch, Wald ...

## Hadamard Condition

Since 1978, a better understanding was successively reached how to characterize the relevant Hilbert space representations of quantum fields in curved spacetimes (for linear quantum fields):

- **Hadamard states** (resp., Hadamard representations) allow a systematic definition of $\langle \mathbf{T}_{\mu\nu}(x) \rangle$ (Wald 1978)

- Hawking effect appears in a natural manner in Hadamard representations (Haag, Narnhofer u. Stein 1984; Fredenhagen u. Haag 1990; Kay u. Wald 1991)

- Hadamard states define a unique Hilbert space representation (Verch 1994)

Hadamard condition on 2-point correlation:

$$\langle \psi | \Phi(x)\Phi(y) | \psi \rangle = \frac{U(x,y)}{\sigma(x,y)} + V(x,y)\ln(\sigma(x,y)) + W(x,y)$$

$\sigma(x,y)$ = squared geodesic distance between $x$ and $y$

## Microlocal Spectrum Condition ($\mu$SC)

An important step was the introduction of the

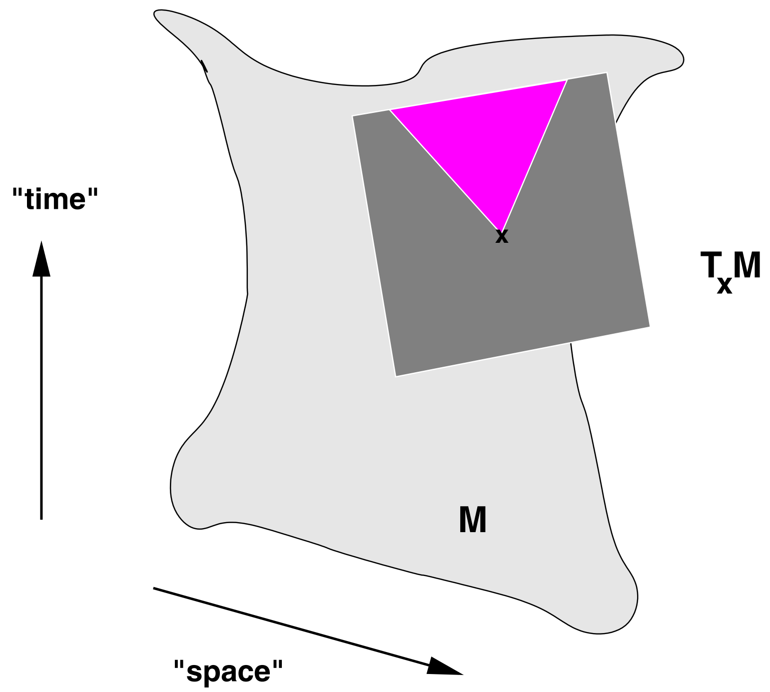
**microlocal spectrum condition** by Radzikowski 1996; Brunetti, Fredenhagen and Köhler 1996.

The microlocal spectrum condition for a state vector $|\psi\rangle$ requires that

$$\widehat{(\chi\Phi)}(k)|\psi\rangle \sim \frac{1}{|k|^N} \quad \forall\ N\ (|k| \to \infty)\,,$$

if $k \in T^*_x M$ is <u>not</u> contained in the (dual) forward light cone of $x \in M$, with test-fuction $\chi$ concentrated around $x$.

(It says that $WF(f \mapsto \Phi(f)|\psi\rangle)$ is contained in the forward light cone bundle)



The microlocal spectrum condition was shown to be equivalent to the Hadamard condition by Radzikowski 1996:

For a linear quantum field $\Phi$ in a Hilbert space representation,

$$\langle\psi|\Phi(x)\Phi(y)|\psi\rangle \quad \text{is Hadamard}$$

$$\Longleftrightarrow$$

$$|\psi\rangle \quad \text{fulfills} \quad \mu\text{SC}$$

The $\mu$SC is more general than the Hadamard condition since it can be generalized to nonlinear quantum fields,

by imposing fall-off conditions on expressions of the form

$$\widehat{(\chi_1\Phi)}(k_1)\cdots\widehat{(\chi_n\Phi)}(k_n)|\psi\rangle$$

for $|k_1| + \cdots + |k_n| \to \infty$ outside of certain conic sets

This was used by Brunetti and Fredenhagen for the perturbative construction of interacting quantum field theories in curved spacetimes — see below.

The $\mu$SC can be seen as a short distance/high energy remnant of the spectrum condition in combination with the equivalence principle

## Quantum Energy Inequalities (QEIs)

In 1978, L. Ford introduced another condition for "admissible" Hilbert space representations of quantum fields on curved spacetimes:

They should satisfy **quantum energy inequalities**:

- ☐ for every timelike curve $\gamma$
- ☐ for every positive $C^\infty$ weight function $f$

there should be a bound of the form

$$\min_{|\psi\rangle} \int_\gamma d\tau\, f(\tau)\langle\psi|\mathbf{T}_{00}(\tau)|\psi\rangle \geq -c_{\gamma,f} > -\infty$$

**Interpretation**: When averaging over finite time, it is impossible to extract an arbitrary amount of energy from any state.

**Note**: The classical **pointwise** weak energy energy condition

$T_{00}(x) = T_{\mu\nu}(x)t^\mu t^\nu \geq 0$ for all timelike vectors $t^\mu$ at $x \in M$

is **violated** in quantum field theory (also on Minkowski spacetime); it holds that

$$\min_{|\psi\rangle} \langle\psi|\mathbf{T}_{00}(x)|\psi\rangle = -\infty \quad !$$

Thus, the QEIs impose a nontrivial constraint on Hilbert space representations to be admissible.

## Relations Between the Conditions: Equivalence Results

For Hilbert space representations of linear quantum fields (Klein-Gordon, Dirac, Maxwell) on generic spacetime manifolds, it could be shown (Fewster 2000; Fewster and Verch 2001; Fewster and Pfenning 2003) that

$$\boldsymbol{\mu SC} \quad \Longrightarrow \boldsymbol{QEIs}$$

For linear quantum fields on static spacetimes, it was found that the following conditions on their Hilbert space representations are equivalent (Fewster and Verch 2002):

$\boldsymbol{\mu SC}$ "microscopic condition"

$\Longleftrightarrow$ $\boldsymbol{QEIs}$ "mesoscopic condition"

$\Longleftrightarrow$ existence of thermal equilibrium states (passive states)

"macroscopic condition"

This shows that $\mu$SC and QEIs can be viewed as equivalent characterizations of quantum field states (or Hilbert space representations) which are dynamically stable — they also coincide with the usual characterizations of the "correct" Hilbert space representations when the spacetime admits time-symmetries.

We will soon see other consequences of imposing these conditions in quantum field theory in curved spacetimes.

## Local General Covariance

In General Relativity, “spacetime” is not a priori given, but needs to be dynamically determined, while observing the principle of general covariance.

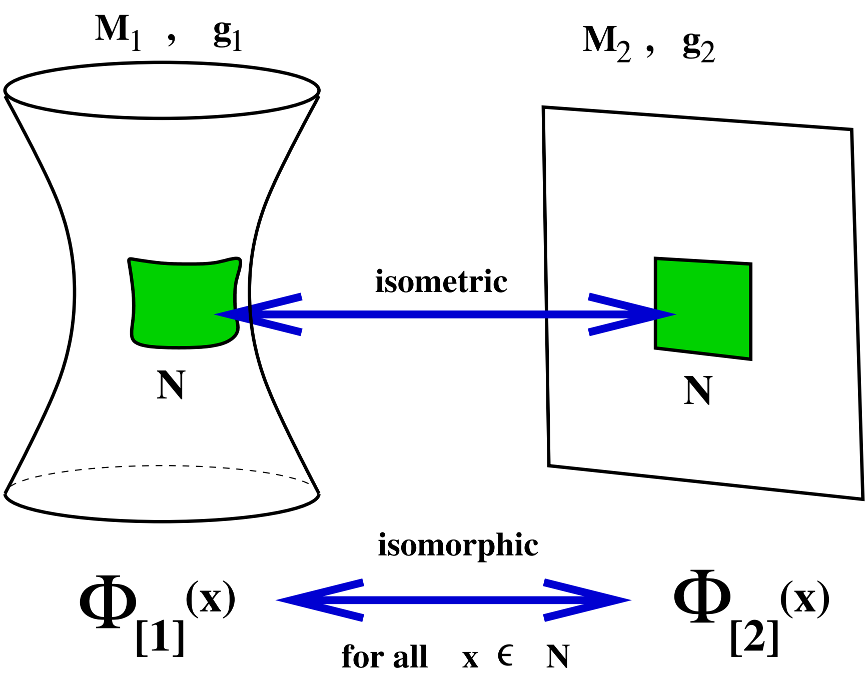
The recent formulation of this principle in QFT in CST is like this (Verch 2001; Hollands and Wald 2001; Brunetti, Fredenhagen and Verch 2003):

**Local general covariance**

(1) To every spacetime $(M, g)$, a quantum field is assigned:

$$(M, g) \longrightarrow \Phi_{[M,g]}(x) \quad \text{quantum field on } (M, g)$$

(2) If two spacetimes have isometric subregions, then the Hilbert space representations of the corresponding quantum fields (restricted to the subregions) have to be isomorphic.



## Spin and Statistics and PCT for Generally Covariant QFT

Let

$$(M, g) \longrightarrow \Phi_{[M,g]}$$

be a quantum field on curved spacetimes fulfilling local general covariance.

(I) **Spin and Statistics** (Verch 2001):

Suppose that the quantum field fulfills the Wightman axioms on Minkowski spacetime and obeys a causal dynamical law. Then

$\Phi_{[M,g]}$ has the correct relation between spin and statistics on each $(M, g)$ :

— if $\Phi_{[M,g]}$ has integer spin, it is bosonic

— if $\Phi_{[M,g]}$ has half-integer spin, it is fermionic.

(II) **PCT** (Hollands 2003):

Suppose that the quantum field fulfills (a strong form of) $\mu$SC and admits an operator product expansion around each point in spacetime.

Then for each given spacetime there is an anti-linear operator relating the operator product expansion of the quantum field on the given spacetime with the operator product expansion of the conjugate-charged quantum field on the same spacetime, but with the reversed spacetime-orientation.

Linear quantum fields of fixed type (e.g Dirac, Proca..) in $\mu$SC representations are examples for local generally covariant quantum fields

## $\mu$SC + Local General Covariance
## $\Rightarrow$ Renormalized Perturbation Theory of $P(\Phi)_4$ on CST

In order to study interacting quantum fields — here, the scalar field with $P(\Phi)_4$ self-interaction — one starts with the free scalar Klein-Gordon field

$$\Phi = \Phi_{[M,g]} \quad \text{on the spacetime } (M, g)$$

in a $\mu$SC Hilbert space representation and then tries to define

- **normal ordered products**

$$\mathcal{N}_n(\Phi(x_1)\cdots\Phi(x_n)) \,, \quad \text{and}$$

- **time ordered products**

$$\mathcal{T}_n(\Phi(x_1)\cdots\Phi(x_n))$$

  of the field operators to all orders $n$, as well as time-ordered products of normal ordered products, and so on.

When this is achieved, then the effects of a given $P(\Phi)_4$ (self-) interaction can be calculated to all orders of perturbation theory using suitable combinations of these expressions.

This program was recently successfully implemented in quantum field theory in curved spacetime:

**Infinite Renormalization** (Brunetti and Fredenhagen 2000):

Brunetti and Fredenhagen generalized the Stueckelberg-Shirkov-Epstein-Glaser approach to renormalizing selfinteracting quantum fields to curved spacetime.

Using $\mu$SC, they showed that the $\mathcal{N}_n$, $\mathcal{T}_n$... can be defined inductively by a consistent prescription extracting finite parts of their singularities at coinciding spacetime points.

The $\mathcal{N}_n$, $\mathcal{T}_n$... are then defined up to smooth parts (renormalization ambiguity). The renormalizability criteria are the same as the power-counting criteria on Minkowski spacetime.

**Reduction of the Renormalization Ambiguity and General Covariance** (Hollands and Wald 2001, 2002):

Hollands and Wald showed that the normal ordering and time ordering prescriptions can be implemented such that

$$(M,g) \longrightarrow \mathcal{N}_{n[M,g]}\,, \quad (M,g) \longrightarrow \mathcal{T}_{n[M,g]}\,, \quad \text{etc}$$

fulfill the principle of local general covariance.

Then the remaining ambiguity in the definition of these quantities is up to only finitely many parameters for each order of perturbation theory (i.e., for each $n$):

$$\tilde{\mathcal{T}}_n(x_1,\ldots,x_n) = \mathcal{T}_n(x_1,\ldots,x_n) + \mathcal{P}_n(x_1,\ldots,x_n)$$

where $\mathcal{P}_n$ is a (known) polynomial in the $\mathcal{N}_k$ and curvature quantities.

## Exotic Spacetimes as Solutions to Semiclassical Gravity?

Spacetimes admitting **closed causal curves** correspond to the idea of **time travel** or **time machines**

*Examples:*

- Spacetimes with "rolled up time-axis" (e.g., Anti-de Sitter)
- Gödel's Universe
- spacetimes with extended, cylindrical, rotating masses (Tipler 1974)
- Spacetimes with "Cauchy-Horizons" – correspond to the idea that a time machchine can be "switched on" (Hawking 1991)

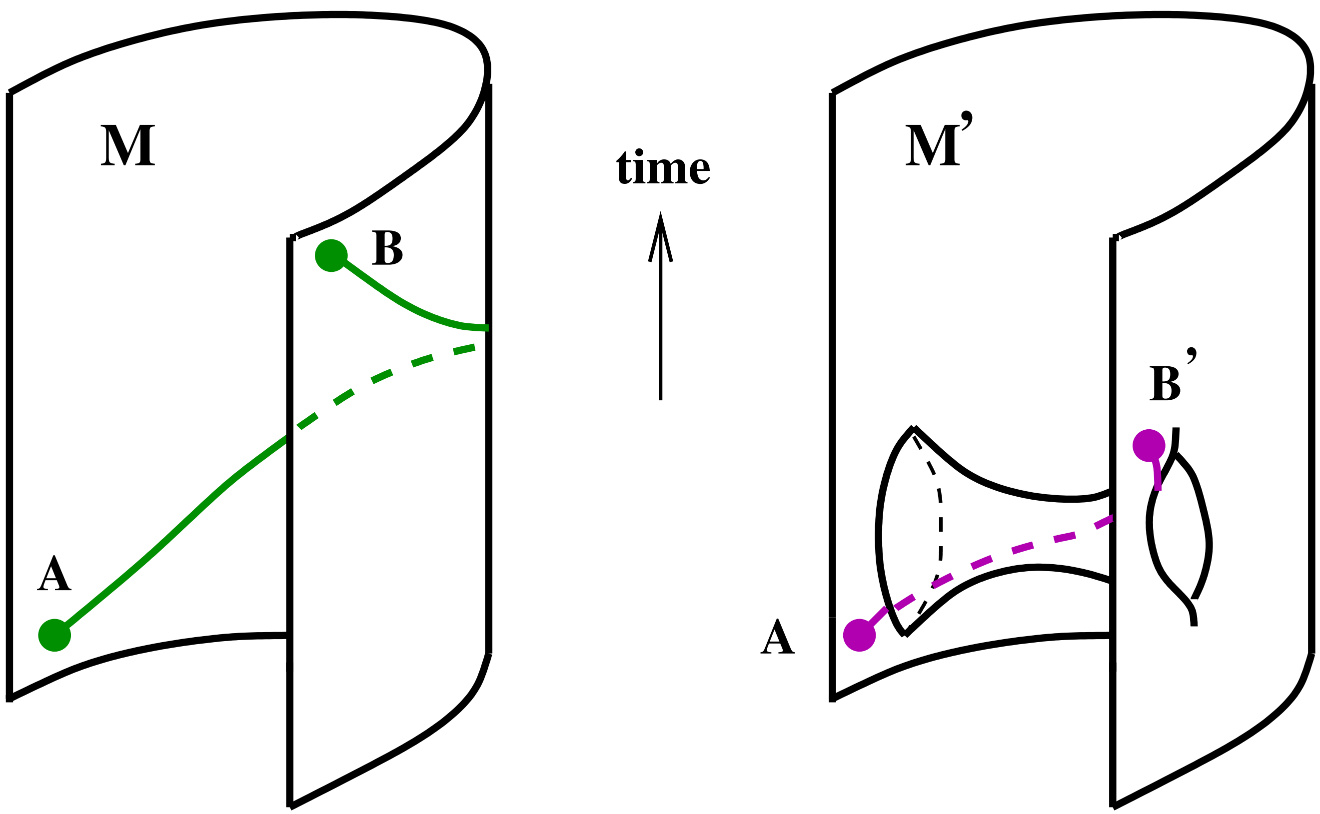
A related class of spacetime scenarios are those admitting **superluminal travel**

*Examples:*

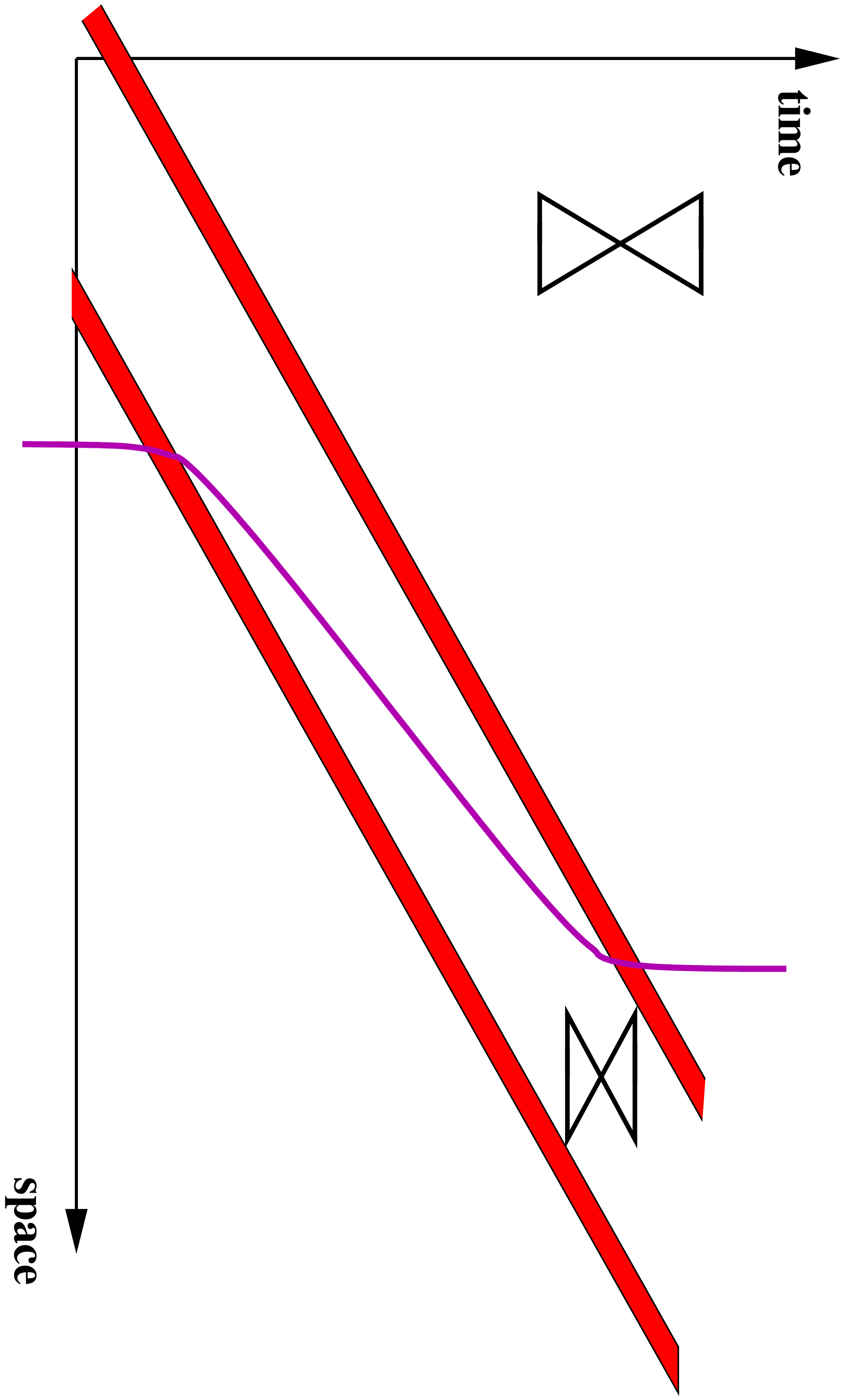
- "wormholes" connecting distant parts of spacetime by a "tunnel" (Morris u. Thorne 1988)
- "warp drive" spacetimes, where the spacetime metric is deformed "externally" and "faster than light" (Alcubierre 1994)

Such scenarios give rise to all kinds of paradoxes.

# faster travel by wormhole (Morris, Thorne, Yurtsever (1988))



## faster travel by "warp–drive" (Alcubierre (1994))



For solutions to the **classical Einstein equations**, these scenarios are severely constrained (essentially, excluded) by the **dynamical stability conditions** for classical matter, such as the weak energy condition:

$$T_{\mu\nu}(x)t^\mu t^\nu \geq 0\,, \quad t^\mu \text{ timelike}$$

As pointed out before, such **pointwise** positivity conditions for the energy are violated in quantum field theory:

$$\langle \mathbf{T}_{\mu\nu}(x)\rangle t^\mu t^\nu$$

can – at each single spacetime point $x$ – attain arbitrarily positive as well as **negative** values!

Observing this, will there appear solutions to the equations of **semiclassical gravity**,

$$R_{\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}(x)R(x) = -\frac{8\pi \mathrm{G}}{c^2}\langle \mathbf{T}_{\mu\nu}(x)\rangle\,,$$

in which "closed timelike curve scenarios" or "superluminal travel scenarios" occur — maybe even generically?

Alternatives:

**Yes!** (Kip Thorne) vs. **No!** (Stephen Hawking)

**Results on this case:**

⊖ **time machines** (Kay, Radzikowski and Wald 1997)

For quantum field theories in Hilbert space representations fulfilling $\mu$SC and existence of a causal dynamical law, spacetimes with Cauchy-horizons are excluded as solutions to the equations of semiclassical gravity.

⊖ **warp drive** (Pfenning and Ford 1997)

For quantum fields in Hilbert space representations fulfilling QEIs, extreme amounts of negative energy (comparable to the total energy of the luminous universe) would have to be concentrated in microscopic domains of space.

⊖ **wormholes** (Ford and Roman 1995)

Again, for quantum fields in Hilbert space representations fulfilling QEIs, extreme amounts of negative energy (comparable to the total energy of the luminous universe) would have to be concentrated in microscopic domains of space in order to sustain macroscopic wormholes. (The situation for microscopic wormholes is not completely clarified.)

## Summary

Most of the central results known for QFT on Minkowski spacetime can now also be obtained in QFT on CST — despite the lack of spacetime symmetries and the related lack of special states like the vacuum state.

- The correct Hilbert space representations of quantum fields can be characterized by $\mu$SC or QEIs

- $\mu$SC and QEIs coincide with the usual requirement of existence of thermal equilibrium or vacuum states in the presence of spacetime symmetries (dynamical stability)

- Local general covariance is a powerful principle which allows it to derive model-independent, structural results like spin-statistics or PCT

- Complete renormalization program for $P(\phi)_4$ interactions compatible with dynamical stability and local general covariance

- Principles of dynamical stability ($\mu$SC, QEIs) exclude exotic spacetime scenarios as solutions to the equations of semiclassical gravity