Unified Field Theory of Gravitation and Electricity

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Among the theoretical physicists working in the field of the general theory of relativity there should be a consensus about the consubstantiality of the gravitational and electromagnetic field. However, I was not able to succeed in finding a convincing formulation of this connection so far. Even in my article published in these session reports (XVII, p. 137, 1923) which is entirely based on the foundations of Eddington, I was of the opinion that it does not reflect the true solution of this problem. After searching ceaselessly in the past two years I think I have now found the true solution. I am going to communicate it in the following.

The applied method can be characterized as follows. First, I looked for the formally most simple expression for the law of gravitation in the absence of an electromagnetic field, and then the most natural generalization of this law. This theory appeared to contain Maxwell's theory in first approximation. In the following I shall outline the scheme of the general theory (§ 1) and then show in which sense this contains the law of the pure gravitational field (§ 2) and Maxwell's theory (§ 3).

§ 1. The general theory

We consider a 4-dimensional continuum with an affine connection, i.e. a $\Gamma^{\mu}_{\alpha\beta}$-field which defines infinitesimal vector shifts according to the relation

$$dA^{\mu} = -\Gamma^{\mu}_{\alpha\beta} A^{\alpha} dx^{\beta}. \tag{1}$$

We do not assume symmetry of the $\Gamma^{\mu}_{\alpha\beta}$ with respect to the indices α and β. From these quantities Γ we can derive the Riemannian tensors

$$R^{\alpha}_{\mu.\nu\beta} = -\frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x_{\beta}} + \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\mu\beta} + \frac{\partial \Gamma^{\alpha}_{\mu\beta}}{\partial x_{\nu}} + \Gamma^{\sigma}_{\mu\nu}\Gamma^{\alpha}_{\sigma\beta}$$

and

$$R_{\mu\nu} = R^{\alpha}_{\mu.\nu\alpha} = -\frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x_{\alpha}} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\alpha\nu} + \frac{\partial \Gamma^{\alpha}_{\mu\alpha}}{\partial x_{\nu}} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} \tag{2}$$

in a well-known manner. Independently from this affine connection we introduce a contravariant tensor density $\mathfrak{g}^{\mu\nu}$, whose symmetry properties we leave undetermined as well. From both quantities we obtain the scalar density

$$\mathfrak{H} = \mathfrak{g}^{\mu\nu} R_{\mu\nu} \tag{3}$$

and postulate that all the variations of the integral

$$\mathcal{J} = \int \mathfrak{H}\; dx_1 dx_2 dx_3 dx_4$$

with respect to the $\mathfrak{g}^{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu}$ as independent (i.e. not to be varied at the boundaries) variables vanish.

The variation with respect to the $\mathfrak{g}^{\mu\nu}$ yields the 16 equations

$$R_{\mu\nu} = 0, \tag{4}$$

the variation with respect to the $\Gamma^{\alpha}_{\mu\nu}$ at first the 64 equations

$$\frac{\partial \mathfrak{g}^{\mu\nu}}{\partial x_\alpha} + \mathfrak{g}^{\beta\nu}\Gamma^{\mu}_{\beta\alpha} + \mathfrak{g}^{\mu\beta}\Gamma^{\nu}_{\alpha\beta} - \delta^{\nu}_{\alpha}\Big(\frac{\partial \mathfrak{g}^{\mu\beta}}{\partial x_\beta} + \mathfrak{g}^{\sigma\beta}\Gamma^{\mu}_{\sigma\beta}\Big) - \mathfrak{g}^{\mu\nu}\Gamma^{\beta}_{\alpha\beta} = 0. \tag{5}$$

We are going to begin with some considerations that allow us to replace the eqns. (5) by simpler ones. If we contract the l.h.s. of (5) by ν and α or μ and α, we obtain the equations

$$3\Big(\frac{\partial \mathfrak{g}^{\mu\alpha}}{\partial x_\alpha} + \mathfrak{g}^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta}\Big) + \mathfrak{g}^{\mu\alpha}(\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\beta}_{\alpha\beta}) = 0. \tag{6}$$

$$\frac{\partial \mathfrak{g}^{\mu\alpha}}{\partial x_\alpha} - \frac{\partial \mathfrak{g}^{\alpha\nu}}{\partial x_\alpha} = 0. \tag{7}$$

If we further introduce the quantities $\mathfrak{g}_{\mu\nu}$ which are the normalized subdeterminants of the $\mathfrak{g}^{\mu\nu}$ and thus fulfill the equations

$$\mathfrak{g}_{\mu\alpha}\mathfrak{g}^{\nu\alpha} = \mathfrak{g}_{\alpha\mu}\mathfrak{g}^{\alpha\nu} = \delta^{\nu}_{\mu}.$$

and if we now multiply (5) by $\mathfrak{g}_{\mu\nu}$, after pulling up one index the result may be written as follows:

$$2\mathfrak{g}^{\mu\alpha}\Big(\frac{\partial\, \lg\sqrt{\mathfrak{g}}}{\partial x_\alpha} + \Gamma^{\beta}_{\alpha\beta}\Big) + (\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\beta}_{\beta\alpha}) + \delta^{\nu}_{\mu}\Big(\frac{\partial \mathfrak{g}^{\beta\alpha}}{\partial x_\alpha} + \mathfrak{g}^{\sigma\beta}\Gamma^{\beta}_{\sigma\beta}\Big) = 0, \tag{8}$$

while γ denotes the determinant of $\mathfrak{g}_{\mu\nu}$. The equations (6) and (8) we write in the form

$$\mathfrak{f}^{\mu} = \frac{1}{3}\mathfrak{g}^{\mu\alpha}(\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\beta}_{\beta\alpha}) = -\Big(\frac{\partial \mathfrak{g}^{\mu\alpha}}{\partial x_\alpha} + \mathfrak{g}^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta}\Big) = -\mathfrak{g}^{\mu\alpha}\Big(\frac{\partial\, \lg\sqrt{\mathfrak{g}}}{\partial x_\alpha} + \Gamma^{\beta}_{\alpha\beta}\Big), \tag{9}$$

whereby $\mathfrak{f}^{\mu}$ stands for a certain tensor density. It is easy to prove that the system (5) is equivalent to the system

$$\frac{\partial \mathfrak{g}^{\mu\nu}}{\partial x_\alpha} + \mathfrak{g}^{\beta\nu}\Gamma^{\mu}_{\beta\alpha} + \mathfrak{g}^{\mu\beta}\Gamma^{\nu}_{\alpha\beta} - \mathfrak{g}^{\mu\nu}\Gamma^{\beta}_{\alpha\beta} + \delta^{\nu}_{\alpha}\mathfrak{f}^{\mu} = 0 \tag{10}$$

in conjunction with (7). By pulling down the upper indices we obtain the relations

$$\mathfrak{g}_{\mu\nu} = \frac{g_{\mu\nu}}{\sqrt{-g}} = g_{\mu\nu}\sqrt{-g},$$

whereby$\mathfrak{g}_{\mu\nu}$ is a covariant tensor

$$-\frac{\partial g_{\mu\nu}}{\partial x_\alpha} + g_{\sigma\nu}\Gamma^{\sigma}_{\mu\alpha} + g_{\mu\sigma}\Gamma^{\sigma}_{\alpha\nu} + g_{\mu\nu}\phi_\alpha + g_{\mu\alpha}\phi_\nu = 0, \tag{10a}$$

whereby ϕ_τ is a covariant vector. This system, together with the two systems given above,

$$\frac{\partial \mathfrak{g}^{\nu\alpha}}{\partial x_\alpha} - \frac{\partial \mathfrak{g}^{\alpha\nu}}{\partial x_\alpha} = 0 \tag{7}$$

and

$$0 = R_{\mu\nu} = -\frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x_\alpha} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\alpha\nu} + \frac{\partial \Gamma^{\alpha}_{\mu\alpha}}{\partial x_\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}, \tag{4}$$

are the result of the variational principle in the most simple form. Looking at this result, it is remarkable that the vector ϕ_τ occurs besides the tensor $(\mathfrak{g}_{\mu\nu})$ and the quantities $\Gamma^{\alpha}_{\mu\nu}$. To obtain consistency with the known laws of gravitation end electricity, we have to interpret the symmetric part of $\mathfrak{g}_{\mu\nu}$ as metric tensor and the skew-symmetric part as electromagnetic field, and we have to assume the vanishing of ϕ_τ, which will be done in the following. For a later analysis (e.g. the problem of the electron), we will have to keep in mind that the Hamiltonian principle does not indicate a vanishing ϕ_τ. Setting ϕ_τ to zero leads to an overdetermination of the field, since we have $16 + 64 + 4$ algebraically independent differential equations for $16 + 64$ variables.

§ 2. **The pure gravitational field as special case**

Let the $g_{\mu\nu}$ be symmetric. The equations (7) are fulfilled identically. By changing μ to ν in (10a) and subtraction we obtain in easily understandable notation

$$\Gamma_{\nu,\mu\alpha} + \Gamma_{\mu,\alpha\nu} - \Gamma_{\mu,\nu\alpha} - \Gamma_{\nu,\alpha\mu} = 0. \tag{11}$$

If Δ is called the skew-symmetric part of Γ with respect to the last two indices, (11) takes the form

$$\Delta_{\nu,\mu\alpha} + \Delta_{\mu,\alpha\nu} = 0$$

or

$$\Delta_{\nu,\mu\alpha} = \Delta_{\mu,\nu\alpha}. \tag{11a}$$

This symmetry property of the first two indices contradicts the antisymmetry of the last ones, as we learn from the series of equations

$$\Delta_{\mu,\nu\alpha} = -\Delta_{\mu,\alpha\nu} = -\Delta_{\alpha,\mu\nu} = \Delta_{\alpha,\nu\mu} = \Delta_{\nu,\alpha\mu} = -\Delta_{\nu,\mu\alpha}.$$

This, in conjunction with (11a), compels the vanishing of all Δ. Therefore, the Γ are symmetric in the last two indices as in RIEMANNian geometry. The equations (10a) can be resolved in a well-known manner, and one obtains

$$\frac{1}{2} g^{\alpha\beta}\Big(\frac{\partial g_{\mu\beta}}{\partial x_\nu} + \frac{\partial g_{\nu\beta}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\beta}\Big). \tag{12}$$

Equation (12), together with (4) is the well-known law of gravitation. Had we presumed the symmetry of the $g_{\mu\nu}$ at the beginning, we would have arrived at (12) and (4) directly. This seems to be the most simple and coherent derivation of the gravitational equations for the vacuum to me. Therefore it should be seen as a natural attempt to encompass the law of electromagnetism by generalizing these considerations rightly. Had we not assumed the vanishing of the ϕ_τ, we would have been unable to derive the known law of the gravitational field in the above manner by assuming the symmetry of the $g_{\mu\nu}$. Had we assumed the symmetry of both the $g_{\mu\nu}$ *and* the $\Gamma^\alpha_{\mu\nu}$ instead, the vanishing of ϕ_α would have been a consequence of (9) or (10a) and (7); we would have obtained the law of the pure gravitational field as well.

§ 3. **Relations to MAXWELL's theory**

If there is an electromagnetic field, that means the $\mathfrak{g}^{\mu\nu}$ or the $g_{\mu\nu}$ do contain a skew-symmetric part, we cannot solve the eqns. (10a) any more with respect to the $\Gamma^\alpha_{\mu\nu}$, which significantly complicates the clearness of the whole system. We succeed in resolving the problem however, if we restrict ourselves to the first approximation. We shall do this and once again postulate the vanishing of ϕ_τ. Thus we start with the ansatz

$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu} + \phi_{\mu\nu}, \tag{13}$$

whereby the $\gamma_{\mu\nu}$ should be symmetric, and the $\phi_{\mu\nu}$ skew-symmetric, both should be infinitely small in first order. We neglect quantities of second and higher orders. Then the $\Gamma^\alpha_{\mu\nu}$ are infinitely small in first order as well.

Under these circumstances the system (10a) takes the more simple form

$$+\frac{\partial g^{\mu\nu}}{\partial x_\alpha} + \Gamma^\nu_{\mu\alpha} + \Gamma^\mu_{\alpha\nu} = 0. \tag{10b}$$

After applying two cyclic permutations of the indices μ, ν and α two further equations appear. Then, out of the three equations we may calculate the Γ in a similar manner as in the symmetric case. One obtains

$$-\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\Big(\frac{\partial g_{\alpha\nu}}{\partial x_\mu} + \frac{\partial g_{\mu\alpha}}{\partial x_\nu} - \frac{\partial g_{\nu\mu}}{\partial x_\alpha}\Big). \tag{14}$$

Eqn. (4) is reduced to the first and third term. If we put the expression $\Gamma^{\alpha}_{\mu\nu}$ from (14) therein, one obtains

$$-\frac{\partial^2 g_{\nu\mu}}{\partial x_\alpha^2}+\frac{\partial^2 g_{\alpha\mu}}{\partial x_\nu \partial x_\alpha}+\frac{\partial^2 g_{\alpha\nu}}{\partial x_\mu \partial x_\alpha}-\frac{\partial^2 g_{\alpha\alpha}}{\partial x_\mu \partial x_\nu}=0. \tag{15}$$

Before further consideration of (15), we develop the series from equation (7). Firstly, out of (13) follows that the approximation we are interested in yields

$$\mathfrak{g}^{\mu\nu}=-\delta_{\mu\nu}-\gamma_{\mu\nu}+\phi_{\mu\nu}, \tag{16}$$

Regarding this, (7) transforms to

$$\frac{\partial \phi_{\mu\nu}}{\partial x_\nu}=0. \tag{17}$$

Now we put the expressions given by (13) into (15) and obtain with respect to (17)

$$-\frac{\partial^2 \gamma_{\mu\nu}}{\partial x_\alpha^2}+\frac{\partial^2 \gamma_{\mu\alpha}}{\partial x_\nu \partial x_\alpha}+\frac{\partial^2 \gamma_{\nu\alpha}}{\partial x_\mu \partial x_\alpha}-\frac{\partial^2 \gamma_{\alpha\alpha}}{\partial x_\mu \partial x_\nu}=0 \tag{18}$$

$$\frac{\partial^2 \phi_{\mu\nu}}{\partial x_\alpha^2}=0. \tag{19}$$

The expressions (18), which may be simplified as usual by proper choice of coordinates, are the same as in the absence of an electromagnetic field. In the same manner, the equations (17) and (19) for the electromagnetic field do not contain the quantities $\gamma_{\mu\nu}$ which refer to the gravitational field. Thus both fields are - in accordance with experience - independent in first approximation.
The equations (17), (19) are nearly equivalent to MAXWELL's equations of empty space. (17) is one MAXWELLian system. The expressions

$$\frac{\partial \phi_{\mu\nu}}{\partial x_\alpha}+\frac{\partial \phi_{\nu\alpha}}{\partial x_\mu}+\frac{\partial \phi_{\alpha\mu}}{\partial x_\nu},$$

which[1] according to MAXWELL should vanish, do not vanish necessarily due to (17) and (19), but their divergences of the form

$$\frac{\partial}{\partial x_\alpha}\Big(\frac{\partial \phi_{\mu\nu}}{\partial x_\alpha}+\frac{\partial \phi_{\nu\alpha}}{\partial x_\mu}+\frac{\partial \phi_{\alpha\mu}}{\partial x_\nu}\Big)$$

however do. Thus (17) and (19) are substantially identical to MAXWELL's equations of empty space.
Concerning the attribution of $\phi_{\mu\nu}$ to the electric and magnetic vectors ($\mathfrak{a}$ and $\mathfrak{h}$) I would like to make a comment that claims validity independently from the theory presented here. According to classical mechanics that uses central forces to every sequence of motion V there is an inverse motion $\bar{V}$, that passes the same configurations by taking an inverse succession. This inverse motion $\bar{V}$ is formally obtained from V by substituting

$$\begin{aligned} x^{'} &= x \\ y^{'} &= y \\ z^{'} &= z \\ t^{'} &= -t \end{aligned}$$

in the latter one.
We observe a similar behavior, according to the general theory of relativity, in the case of a pure gravitational field. To achieve the solution $\bar{V}$ out of V, one has to substitute $t' = -t$ into all field functions and to change the sign of the field components g_{14}, g_{24}, g_{34} and the energy components T_{14},

[1]This appears to be a misprint. The first term should be squared.

T_{24}, T_{34}. This is basically the same procedure as applying the above transformation to the primary motion V. The change of signs in g_{14}, g_{24}, g_{34} and in T_{14}, T_{24}, T_{34} is an intrinsic consequence of the transformation law for tensors.

This generation of the inverse motion by transformation of the time coordinate ($t' = -t$) should be regarded as a general law that claims validity for electromagnetic processes as well. There, an inversion of the process changes the sign of the magnetic components, but not those of the electric ones. Therefore one should have to assign the components ϕ_{23}, ϕ_{31}, ϕ_{12} to the electric field and ϕ_{14}, ϕ_{24}, ϕ_{34} to the magnetic field. We have to give up the inverse assignment which was in use as yet. It was preferred so far, since it seems more comfortable to express the density of a current by a vector rather than by a skew-symmetric tensor of third rank.

Thus in the theory outlined here, (7) respectively (17) is the expression for the law of magnetoelectric induction. In accordance, at the r.h.s. of the equation there is no term that could be interpreted as density of the electric current.

The next issue is, if the theory developed here renders the existence of singularity-free, centrally symmetric electric masses comprehensible. I started to tackle this problem together with Mr. J. Grommer, who was at my disposal ceaselessly for all calculations while analyzing the general theory of relativity in the last years. At this point I would like to express my best thanks to him and to the 'International educational board' which has rendered possible the continuing collaboration with Mr. Grommer.